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Quantum tunnelling, non-trapping and the existence of Stark ladder resonances

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Abstract. We announce results which establish the existence of Stark-Wannier resonances in one-dimension in the semiclassical regime of small Planck's constant. We also obtain estimates on the width of these resonances, including dependence of the width on the electric field in certain cases. We discuss the proofs of these results which utilise many recently developed ideas concerning resonances in quantum mechanics.

Quantum tunnelling has been much discussed in the theoretical physics [1-3] and mathematical physics [4-6] literature. Such ideas apply to the study of stationary states of quantum Hamiltonians $H(h) \equiv h^2 p^2 + V$, $p \equiv -i\nabla$, in the semiclassical regime of small h, when V is a potential having multiple minima separated by barriers of finite height and V becomes infinite as $|x| \to \infty$. Tunnelling also plays a role in the study of quasistationary states of Hamiltonians (see figure 1). There now exist many results on the existence of resonances for two- and many-body Hamiltonians in the generalised semiclassical regime. These Hamiltonians include those describing shape resonances [7-9], the DC and AC Stark effect for two- and many-body systems [10-13], and the Zeeman effect [8]. The purpose of this paper is twofold. First, we announce results establishing the existence of Stark ladder resonances in one dimension and estimates on the width of the resonances, including a proof of an Oppenheimer-type formula. The details will be published elsewhere [14]. Secondly, we use this model to explain the key ideas concerning the results on quantum resonances mentioned above. The problem of Stark ladder resonances has attracted considerable debate in the physics literature [15-17] and the solution of this problem (at least in the semiclassical regime) involves all of the characteristic ideas recently developed about quantum resonances: spectral deformation, geometric perturbation theory and exponential decay estimates of eigenfunctions.

The notion of a resonance for a quantum system is based on phenomenology but is difficult to formulate mathematically within the usual framework of quantum mechanics. We use a definition which originated in the work of Aguilar, Balslev and Combes [18, 19]. Let H be the self-adjoint Hamiltonian for the system on a Hilbert space $\mathcal{H}$. Our definition involves the resolvent or Green function $R(z)$ of H and a unitary group $U(\theta)$, $\theta \in \mathbb{R}$. The resolvent $R(z) = (z - H)^{-1}$ is defined and analytic in z on $\mathbb{C} \backslash \sigma(H)$, where $\sigma(H)$ is the spectrum of H (the discrete states and continuous

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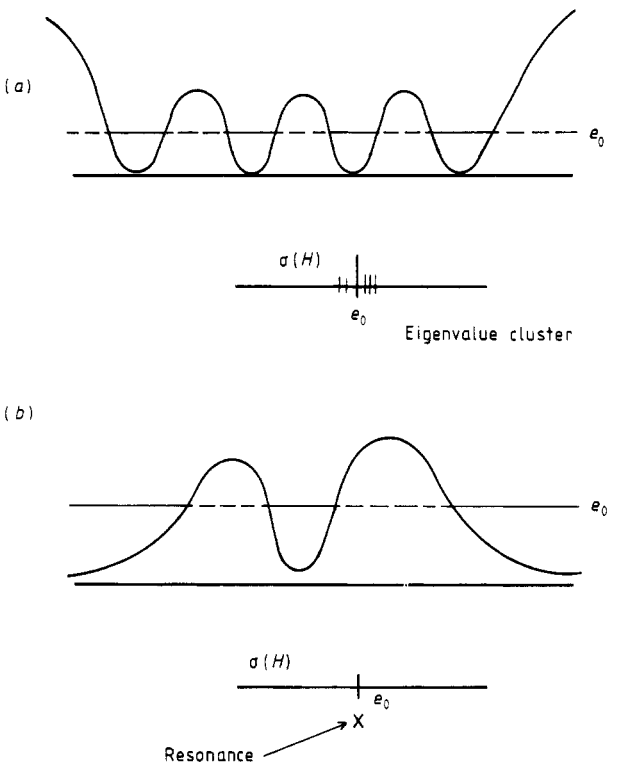

Figure 1. Tunnelling phenomena: (*a*) stationary tunnelling leads to the splitting of eigenvalues; (*b*) dynamic tunnelling leads to resonances.

spectrum). The group $U(\theta)$ must be a *spectral deformation group* for H; that is, the conjugated operator $H(\theta) = U(\theta)\, H U(\theta)^{-1}$, $\theta \in \mathbb{R}$, must admit an analytic continuation in θ to a complex neighbourhood of $\mathbb{R}$. The original choice for $U(\theta)$ was the dilation group on $\mathbb{R}^n : x \to \alpha_\theta(x) \equiv e^{\theta} x$ and for $f \in L^2(\mathbb{R}^n)$:

$$\begin{aligned}(U(\theta)f)(x) &= J_\theta(x)^{1/2} f(\alpha_\theta(x)) \\ &= e^{-\theta n/2} f(e^{\theta} x)\end{aligned} \tag{1}$$

where J_θ is the Jacobian of the map α_θ. One can check the formula for $\theta \in \mathbb{R}$:

$$H(\theta) = U(\theta)(h^2 p^2 + V(x))U(\theta)^{-1} = e^{-2\theta} h^2 p^2 + V(\alpha_\theta(x)) \tag{2}$$

so that if the potential V is relatively p^2-compact and is the restriction to $\mathbb{R}$ of an analytic function $V(z)$ for $|\operatorname{Im} z| < \delta |\operatorname{Re} z|$, then $H(\theta)$ has an analytic continuation into this region $\mathcal{S}$ in general. The group $U(\theta)$ has an associated family of analytic vectors $\mathcal{A}_U$ in $\mathcal{H}$. If $v \in \mathcal{A}_U$, then

$$\theta \in \mathbb{C} \to U(\theta) v$$

is an analytic function. This set $\mathcal{A}_U$ is dense in $\mathcal{H}$. Given H and a spectral deformation group $U(\theta)$, we have definition 1.

Definition 1. A resonance z_0 of a self-adjoint Hamiltonian H is a pole of the meromorphic continuation of matrix elements $\langle u, R(z)v \rangle$, $u, v \in \mathcal{A}_U$ for a spectral deformation group U, from $\operatorname{Im} z > 0$ through the continuous spectrum, to the lower half-complex plane.

Several comments are in order. The first refer to the mathematical nature of definition 1, and the second refer to the connection between definition 1 and the physical notions of resonance. Definition 1 is a mathematically sound definition because:

(i) a resonance z_0 depends only on H and $\mathscr{A}_U$, not on the vectors $u, v \in \mathscr{A}_U$;

(ii) a resonance z_0 is an eigenvalue of the non-self-adjoint operator $H(\theta)$, $\theta \in \mathscr{S}$ (this implies (i));

(iii) a resonance z_0 is locally independent of θ;

(iv) a resonance z_0 is independent of the spectral deformation group in the following sense: let $V(\theta)$, $\theta \in \mathbb{R}$, be another spectral deformation group for H such that $\tilde{H}(\theta) \equiv V(\theta) H V(\theta)^{-1}$ also has a continuation to $\mathscr{S}$ and such that $\mathscr{A}_U \cap \mathscr{A}_V$ is dense in $\mathscr{H}$. Then z_0 is also an eigenvalue of $\tilde{H}(\theta)$.

(v) The group nature of $U(\theta)$ is not necessary; in many cases it suffices that $\theta \to U(\theta)$ for $|\theta|$ small is a unitary family.

A detailed discussion of these facts may be found in [20]. We emphasise that definition 1 is flexible in its implementation (as indicated in (iv)) and useful as it identifies resonances as eigenvalues (albeit of non-self-adjoint operators). This fact is used in both theoretical and numerical calculations. Definition 1 does not assert that a meromorphic continuation of $\langle u, R(z)v\rangle$ exists. Indeed, this requires more detailed information about the spectrum of $H(\theta)$ for $\theta \in \mathbb{C}$. As the name spectral deformation implies, in many cases one can prove that the essential spectrum of H is moved off the real axis and into the lower complex plane as θ becomes complex for $\operatorname{Im}\theta > 0$. In the example above (2), it is easy to check that the continuous spectrum of $H(\theta)$ is the ray $e^{-2i\operatorname{Im}\theta}\mathbb{R}_+$ (see figure 2). This global information permits meromorphic continuation of $\langle u, R(z)v\rangle$ into the sector $\{z \mid -2\operatorname{Im}\theta < \arg z < 0\}$. In many cases, we are only interested in a local continuation for z with $\operatorname{Re} z$ near some energy e_0. The existence of a local meromorphic continuation into a complex neighbourhood of e_0 is then guaranteed by so-called non-trapping estimates [21-23]. This is discussed in more detail below.

With respect to physical interpretation, the resonances so obtained are associated with observable properties of the system. We mention two examples. For $H = p^2 + V$

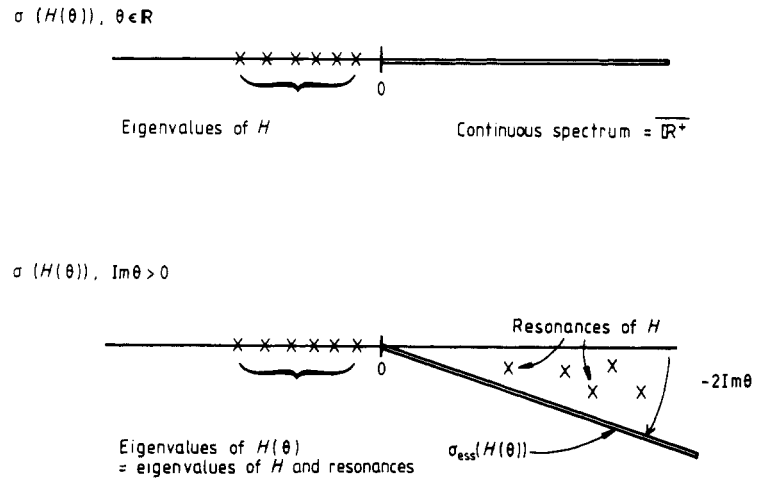

Figure 2. Spectrum of a deformed Hamiltonian.

where V is short-range, i.e, $V(x) = \mathcal{O}(|x|^{-2-\varepsilon})$, $\varepsilon > 0$, and analytic in a neighbourhood of infinity, it is known that the resonances are poles in the meromorphic continuation of the $\mathcal{S}$-matrix $\mathcal{S}(k)$ for the system [24]. This is also true for the two-cluster to two-cluster $\mathcal{S}$-matrix in the N-body case when V is the sum of two-body exponentially decaying potentials [25]. Moreover, the existence of resonances for two-body Schrödinger operators with compactly supported V implies that there are states ψ of the system, well localised in energy and position, such that $\|\chi_R \, e^{iHt}\psi\|$ decays exponentially like $e^{-t/\tau}$ for several lifetimes, where τ is the width of the resonance and V is non-zero for $|x| < R$ [26]. (Strict exponential decay of $e^{iHt}\psi$ is prohibited if $\sigma(H)$ is contained in a half-line. In the Stark effect case, this is not true and there is pure exponential decay [27].)

Finally, we mention another definition of resonance which has been introduced by Helffer and Sjöstrand [8] and which gives results similar to those discussed above. (In fact, it is known that these are equivalent definitions when applied to the Schrödinger equation [28].) Given $H(h)$ one studies $(H(h) - z)$, $z \in \mathbb{C}$, not on the original Hilbert space $\mathcal{H}$, but as a transformation between certain Sobolev spaces which are associated with complex extensions of the classical phase space for the system. For an energy E_0, the resonances of $H(h)$ near E_0 are defined to be the complex z_0 such that $(H(h) - z_0)$ is not invertible.

We now turn to the Stark ladder problem. The Hamiltonian in one dimension with an external electric field $F \geqslant 0$ is:

$$H(h, F) = h^2 p^2 + v + Fx \tag{3}$$

where $p^2 \equiv -d^2/dx^2$ and v is a periodic potential satisfying:

(v1) v is a real, non-constant periodic function with period τ, and v is the restriction to $\mathbb{R}$ of a function $v(z)$ analytic in a strip $|\mathrm{Im}\, z| < \eta$. Furthermore, we must place some restrictions on F relative to h:

(F1) for some $c > 0$, $ch^\alpha < F < \|v'\|_\infty$, where $0 \leqslant \alpha < 1$.

Of course, if $F > 0$ is a constant independent of h, then by taking h small enough this lower bound can always be satisfied. In some cases, F might be a function of h and in these cases, (F1) gives a bound on the h-dependence of F for which our proof works. The upper bound in (F1) guarantees that the total potential $V(x) \equiv Fx + v(x)$ has local maxima and minima which will trap the particle. The lower bound in (F1) ensures that the tunnelling phenomena is controlled by the width in the Agmon metric, defined in (6) below, of the potential barriers in the semiclassical regime of small h. Our first result is the following theorem.

Theorem 1 (Existence). Let $H(h, F)$ be as in (3) with (v1) and (F1) satisfied. Then $\exists h_0 > 0$ such that $h < h_0$ implies that (1) $H(h, F)$ has an infinite family of resonances z_{0j} with

$$\mathrm{Re}\, z_{0,j} = \tilde{e}_0 + jF\tau \qquad j \in \mathbb{Z} \tag{4}$$

and (2) $\mathrm{Im}\, z_{0,j} \equiv \mathrm{Im}\, z_0$. In addition, there exists $c > 0$ and $\beta > 0$ such that:

$$|\mathrm{Im}\, z_0| \leqslant c\, e^{-\beta/h}.$$

The characteristic periodicity of $z_{0,j}$ gives rise to the name Stark ladder resonances. The constant $\tilde{e}_0$ can be identified with the ground state e_0 of a particle confined to a unit cell of the periodic potential v up to an exponentially small correction. This gives the first ladder of resonances. It is expected that ladders of resonances exist for each

of the low-lying eigenvalues of the single-cell Hamiltonian. Other mathematical results concerning (3) include the existence of resonances for F sufficiently large [29] and in the limit $F \to 0$ [30], estimates on the evolution generated by H [31], and on the interband matrix elements [32].

In solid state physics, the existence of Stark–Wannier resonances can be argued as follows. Let $B_\tau \equiv [0, 2\pi/\tau]$ be the Brillouin zone for

$$H_0 = -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + v.$$

Let $\varepsilon_n(k)$, $k \in B_\tau$, $n \in \mathbb{N}$, be the band energy function. H_0 can be diagonalised in a basis of Bloch waves and we use this basis to write $H = H_0 + Fx$ restricted to the nth band. The eigenfunctions $\psi_n(k)$ and eigenvalues E_n of this single-band Hamiltonian satisfy

$$(\varepsilon_n(k) + \mathrm{i}F\partial_k)\psi_n(k) + F \sum_{m \neq n} X_{nm}(k)\psi_m(k) = E_n\psi_n(k)$$

where $X_{nm}(k)$ are the interband matrix elements of the operator x. Neglecting these interband matrix elements, we can solve this equation exactly. The spectrum of H restricted to the nth band is discrete, indexed by $j \in \mathbb{Z}$, and given by

$$E_{nj} = \frac{\tau}{2\pi} \int_0^{2\pi/\tau} \varepsilon_n(k')\, \mathrm{d}(k') + F\tau j.$$

Note that this is similar to (4) for the real part of the resonances. (In fact, $\tilde{e}_0$ is equal to the integral with $n = 0$ up to terms $\mathcal{O}(h^\infty)$). If we now consider the perturbations X_{nm}, these terms couple the energy levels to the continuum and as a result the levels acquire a finite lifetime as indicated by a non-zero imaginary part. In this picture, each band contributes a ladder of resonances.

In our work, we treat the problem of Stark ladder resonances by analysing the tunnelling phenomena as follows (see figure 3). When $F = 0$, the spectrum of $H(h, 0)$ has the well known band structure. Consider dividing $\mathbb{R}$ into unit cells whose length is the period τ of v. We impose Dirichlet boundary conditions between the cells and obtain an approximate Hamiltonian $H_0(h) = \bigoplus_{i=-\infty}^{\infty} h_i(h)$ where all h_i are identical and h_i describes the evolution of a particle trapped in a single cell. The bound states of h_i are $e_0 < e_1 < \ldots$.

Any e_i is an eigenvalue of infinite multiplicity for H_0. We now consider removing the boundary conditions: as $h \to 0$, $H_0(h)$ should approach $H(h, 0)$. This is indeed the case, each infinitely degenerate energy level e_i of $H_0(h)$ splits and gives a band E_i of $H(h)$ [33]. In the semiclassical regime of small h, the width of the band is controlled by the width of the potential barrier

$$\rho_A(e_j) \equiv \frac{1}{h} \int_{x_1}^{x_2} \sqrt{v(s) - e_j}\, \mathrm{d}s \tag{5}$$

which is a distance in the Agmon metric

$$\mathrm{d}s^2 = h^{-2}(V(x) - E)_+\, \mathrm{d}x^2 \tag{6}$$

between two classical turning points x_1 and x_2; $v(x_i) = e_j$. Of course, (5) is familiar from the WKB approximation. The metric (6) however is valid in N dimensions and

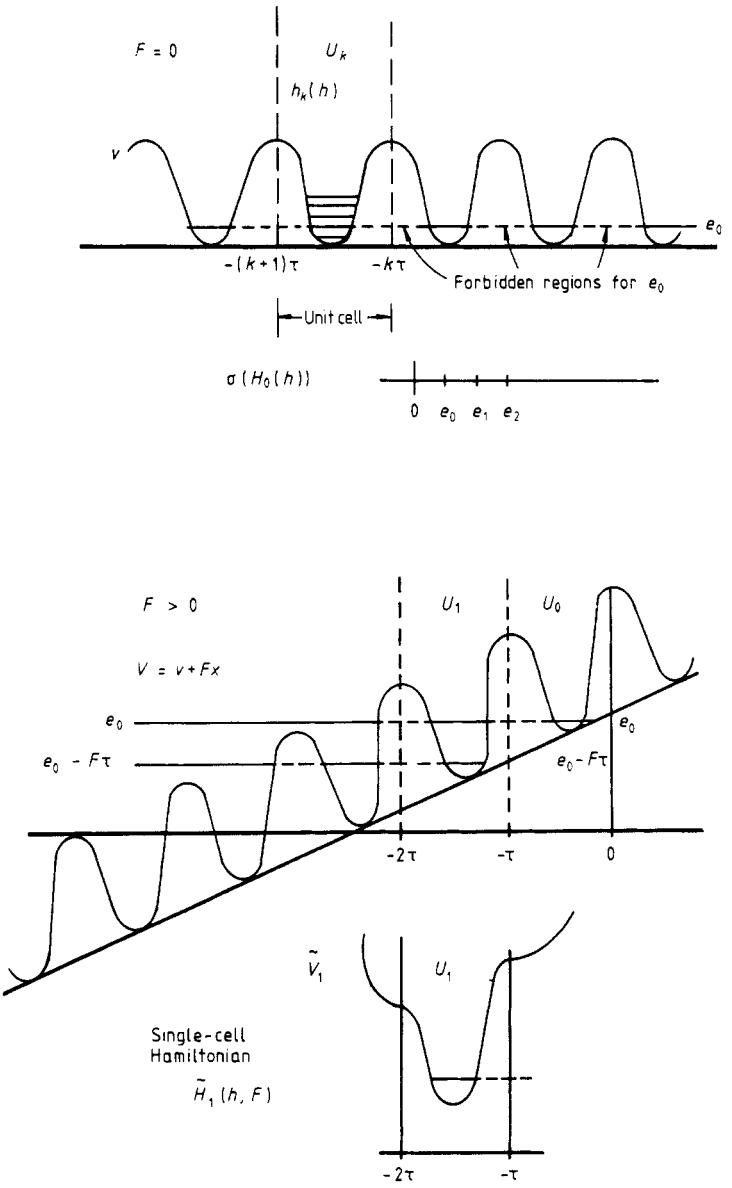

Figure 3. Tunnelling analysis of Stark ladder resonances.

gives the correct N-dimensional analogue of the WKB result. (In fact, the entire discussion of this paragraph holds in N-dimensions.)

We now turn on the electric field F. The spectrum of $H(h, F)$ is dramatically different from $H(h, 0)$: it is the entire real line and there are no eigenvalues! This suggests the second ingredient in our discussion of resonances: the non-trapping property of the potential $V(x) = v(x) + Fx$. We can still consider the unit cells U_i as above but they are no longer identical. The energy levels in U_{i+1} are shifted from those in U_i by an amount $F\tau$. If we consider a particle with energy e_0 in U_0 the force due to the field F will accelerate it to the left. The particle must still tunnel through

the barriers, but the distance, across each barrier, as in (5), is now

$$\frac{1}{h}\int_{x_i}^{x_{i+1}} \sqrt{(Fs+v(s)-c_0)_+}\,\mathrm{d}s \tag{7}$$

where $x_i \in U_i$, $x_{i+1} \in U_{i+1}$ are consecutive turning points. This distance decreases and eventually $e_0 > V(x)$ for all $x < S_T(e_0) \equiv \min\{x \mid V(x) = e_0\}$, the last classical turning point. We expect that for small h the particle behaves classically and, classically, there are no trapped orbits for the Hamiltonian $H(x, p) = h^2p^2 + v(x) + Fx$ at energy e_0 in the region $x < S_T(e_0)$. We discuss a quantum version of this idea below.

Roughly speaking, one expects that the sum of the distances (7) across the barriers,

$$\sum_i \rho_A(U_i, U_{i+1}) \tag{8}$$

determines the lifetime (equal to the inverse of the imaginary part of the resonace) of some of the resonances. This brings us to our second result. To state this, we must discuss the notion of resonant and non-resonant cells at energy e_0 [33, 34]. We fix the x-coordinates (for convenience) such that

$$V(0) = v(0) = v_0 \qquad \text{where } v_0 = \max_{x\in\mathbb{R}} v(x).$$

Then a unit cell for V is an interval $U_k \equiv [-\tau(k+1), -k\tau]$, $k = 0, 1, 2, \ldots$. Associated with U_k is a single-cell Hamiltonian $\tilde{H}_k(h, F) \equiv h^2p^2 + \tilde{V}_k$ on $L^2(\mathbb{R})$ where $\tilde{V}_k | U_k = V$ and $\tilde{V}_k$ is monotone increasing on $\mathbb{R} \setminus U_k$. The energy e_0 in theorem 1 is the ground-state energy of $\tilde{H}_0$ up to corrections $\mathcal{O}(h^\infty)$.

Definition 2. Let e_0 be as in theorem 1 and consider an interval $I_0 \equiv [I_0^-, I_0^+] \ni e_0$ such that $\mathrm{dist}(\sigma(\tilde{H}_0), I_0^\pm) \geq ch^{\alpha+2+\varepsilon}$, for $c > 0$ and for some $\varepsilon > 0$. Then the cell U_k is non-resonant with respect to e_0 if

$$\sigma(\tilde{H}_k) \cap I_0 = \phi. \tag{9}$$

We prove that it is the distance between the cell U_0 and the nearest resonant cell or the exit region Ω which lies in the barrier containing $S_T(e_0)$ and which controls the lifetime of some resonant state with energy near e_0.

Theorem 2 (Width). For each h sufficiently small, there exists a resonance energy z_0 near e_0 and constants $c > 0$, $k > 0$ such that for any $\varepsilon > 0$

$$|\mathrm{Im}\ z_0| \leq c\, \mathrm{e}^{-(k\mathscr{S}_0 - \varepsilon h^{-1})}$$

where

$$\mathscr{S}_0 \equiv \min_{j\in R} \{d_A(U_0, U_j), d_A(\Omega, U_0)\}$$

where R is the set of unit cells resonant with respect to e_0.

This formula permits one to obtain, under additional assumptions, a version of the well known Oppenheimer formula applicable to periodic potentials. The original formula describes the dependence of the resonance width on the field F for a hydrogen atom in an external electric field.

Corollary 1. Suppose that the resonant wells are separated by Euclidean distance $\mathcal{O}(F^{-1})$ or that the distance in the Agmon metric from U_0 to Ω is $\mathcal{O}((Fh)^{-1})$ and there are no other resonant wells than U_0. Then for each h sufficiently small there is a resonance energy z_0 and constants c, $\beta > 0$ such that

$$|\mathrm{Im}\, z_0| \leqslant c\, \mathrm{e}^{-\beta(Fh)^{-1}}. \tag{10}$$

Under condition (F1) it is probably likely that the second option in corollary 1 obtains. If resonances exist for $F < ch^{\alpha}$, $\alpha > 1$, then the first option usually obtains.

We now give a sketch of the main ideas which enter into the proof of theorem 1. The basic idea of the geometric theory of resonances is to introduce an approximate Hamiltonian H_0 which describes a decoupled system and whose spectrum can be easily computed. The decoupling is typically obtained by placing a barrier between the regions where a classical particle of energy E will be trapped and where it will have unbounded orbits. The approximate Hamiltonian H_0 is then a direct sum of the Hamiltonian H_1 describing the evolution in the unbounded region and H_2 describing the motion in the bounded region. For the Stark ladder Hamiltonian (1), this is illustrated in figure 4. The eigenvalues of H_2 are approximations to the resonances of H.

(i) *Approximate Hamiltonian.* We decouple the region W_2 where a particle is classically trapped from the exterior W_1. For energy e_0, let $S_T(e_0) \equiv \min\{x | V(x) = e_0\}$, the last classical turning point. Then the particle is trapped in W_2, i.e. where $V(x) > e_0$ for some x. W_1 is the region where the particle is free, i.e. $V(x) < e_0$ for all x. This decoupling is done in a smooth overlapping way, $W_1 \cap W_2 \equiv \Omega$ is contained in $\mathrm{G}(e_0) \equiv \{x | V(x) \geqslant e_0\}$, the classically forbidden region for energy e_0. We associate a Hamiltonian with each region: $H_1 = h^2p^2 + v(x) + Fx$ on W_1 with a Dirichlet boundary condition at the finite endpoint, and $H_2 = h^2p^2 + V_2(x)$ on $\mathbb{R}$ where $V_2 = V(x)$ for $x \in W_2$ away from Ω and $V_2 \to \infty$ as $x \to -\infty$. In this way, $\sigma(H_1) = \mathbb{R}$ and $\sigma(H_2)$ consists of non-degenerate eigenvalues. In particular, there are eigenvalues near e_0 separated from all other eigenvalues by an amount $\mathcal{O}(h^{2+\alpha+\varepsilon})$, $\varepsilon > 0$. The approximate Hamiltonian is $H_0(h, F) = H_1(h, F) \oplus H_2(h, F)$ on the Hilbert space $\mathcal{H}_0 \equiv L^2(W_1) \oplus L^2(\mathbb{R})$. Note that the spectrum of $H_0(h, F)$ consists of eigenvalues coming from H_2 embedded in the continuum states of H_1. The basic idea of semiclassical resonance theory is that the embedded eigenvalues on H_0 become resonances of H for h small.

(ii) *Quantum non-trapping.* The non-trapping condition on V for a small energy interval about e_0 and in the region $W_{NT} \equiv (-\infty, S_T(e_0) + \delta)$ reflects the fact that there are no bounded classical trajectories with this energy in this region of phase space. This condition can be given in n dimensions but we restrict the definition to $n = 1$. The formulation of this condition requires the introduction of a vector field f: $\mathbb{R} \to \mathbb{R}$ which is increasing along the classical trajectories in the region W_{NT}.

Definition 3. A potential V is non-trapping for the energy interval $I_0 \ni e_0$ and on the region $W_{NT} \subset \mathbb{R}$ if there exist $\varepsilon_0 > 0$ and a vector field $f \in C^1(\mathbb{R})$ such that for all $x \in W_{NT}$ and $E \in I_0$:

$$2f'(x)(E - V(x)) - f(x)V'(x) \geqslant \varepsilon_0 > 0. \tag{11}$$

This formula (11) is a generalisation of the usual quantum virial which one obtains in the case $f(x) = x$. Positivity of the virial usually implies the absence of bound states. Condition (11) will imply the absence of resonances for H_1 in a neighbourhood of e_0. For the Stark ladder problem, construction of f such that (11) is satisfied in the region

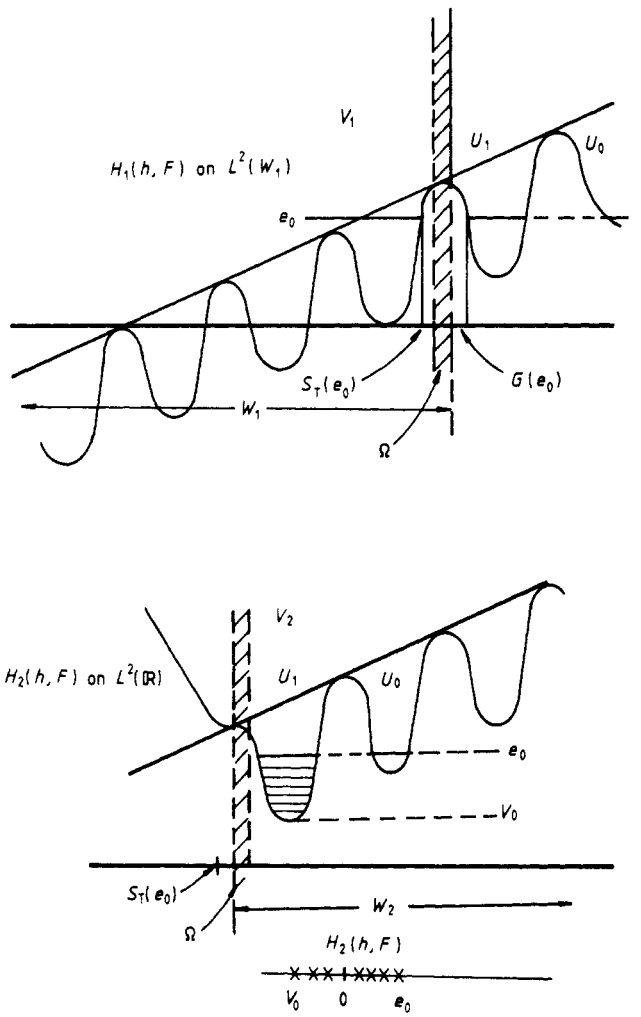

Figure 4. The approximate Hamiltonian $H_0 = H_1(h, F) \oplus H_2(h, F)$.

$W_{NT}(\delta)$ for some $\delta > 0$ requires the patching together of two vector fields f_1 and f_2. For $x \approx S_T(e_0)$,

$$f_1(x) = x$$

and for $x < S_T(I_0^-)$, we take

$$f_2(x) = \frac{1}{2\sqrt{E - V(x)}} \int_{x_0}^{x} \frac{a(s)\,ds}{\sqrt{E - V(s)}}.$$

Note that f_2 is obtained by solving the differential inequality (11). The undetermined function a is chosen so that the smooth vector field

$$f(x) = g_1(x) f_1(x) + g_2(x) f_2(x)$$

satisfies (11). Here g_1 and g_2 are a partition of unity: $g_1 + g_2 = 1$ on W_1 and $S_T(e_0) \in \text{supp } g_2$.

(iii) *Spectral deformation.* According to definition 1 we must associate with H and with H_0 a family of non-self-adjoint Hamiltonians $H(\theta)$ and $H_0(\theta)$, respectively, $\theta \in \mathbb{C}$. This is done in a manner such that the continuous spectrum of H_1 is moved off the real axis but the eigenvalues of H_2 are left intact. We use the vector field f of (ii) to accomplish this. Consider the mapping

$$x \to \alpha_\theta(x) \equiv x + \theta f(x) \qquad \theta \in \mathbb{R}.$$

For $|\theta|$ small, this map preserves W_1 and W_2 and is invertible. We define unitary operators U_θ by

$$(U_\theta g)(x) = (1 + \theta f'(x))^{1/2} g(\alpha_\theta(x))$$

for $|\theta|$ small. By means of U_θ, we define $H_0(\theta)$ and $H(\theta)$ by $U_\theta H_0 U_\theta^{-1}$, $U_\theta H U_\theta^{-1}$, respectively, for $\theta \in \mathbb{R}$. These operators can be analytically continued in θ for $|\text{Im } \theta|$ small and, by Hunziker's work [35], there exists a dense set $\mathscr{A}_U$ of vectors having the properties discussed above. It follows from the non-trapping condition that the continuous spectrum of $H(\theta)$ and $H_0(\theta)$ is deformed off the real axis in a neighbourhood of the eigenvalue of H_2 considered. The resonances of $H(\theta)$ associated with this eigenvalue will lie between this curve and the real axis in the lower-half complex plane.

(iv) *Geometric perturbation theory*: For $\theta \in \mathbb{C}$, $|\theta|$ small, we study the difference of the resolvents $R_0(z, \theta) = (z - H_0(\theta))^{-1}$ and $R(z, \theta) = (z - H(\theta))^{-1}$ in a neighbourhood of $e_0 \in \sigma(H_2)$. We show that for a circle Γ of radius $h^{2+\alpha+\varepsilon}$ about e_0, $R(z, \theta)$ is defined and the difference of the resolvents vanishes as $h \to 0$. This will immediately imply that $H(\theta)$ has an eigenvalue z_0 inside Γ. Since H has no eigenvalues, $\text{Im } z_0 \neq 0$ and z_0 is a resonance of H. The validity of this convergence of resolvents follows from the fact that the difference of the two operators $H(\theta)$ and $H_0(\theta)$ is localised in $\Omega \subset G(e_0)$, i.e. in the classically forbidden region. As h is made small, all wavefunctions localised to this region become small and, consequently, so do the resolvents. The perturbation of $H_0(\theta)$ by $H(\theta)$ is not 'small' in any usual sense, but is localised to a region where the particle is prohibited to be. To prove this, we take a pair of smooth functions J_1 and J_2 such that $J_1^2 + J_2^2 = 1$ and $\text{supp } J_i' \subset \Omega$ and $J_i | (W_i \backslash \Omega) = 1$, i.e. they are 1 outside of Ω. We use J to denote the map

$$J : \mathscr{H}_0 \to \mathscr{H}$$

defined by

$$J(u_1 \oplus u_2) = J_1 u_1 + J_2 u_2.$$

Then it is easy to check that $JJ^* = 1_{\mathscr{H}}$. We can now write the geometric resolvent equation on $\mathscr{H}_0$:

$$R(z, \theta) J = J R_0(z, \theta) + R(z, \theta) M R_0(z, \theta).$$

The interaction term M is given by

$$M(u_1 \oplus u_2) = h^2 [p^2, J_1] u_1 + h^2 [p^2, J_2] u_2$$

which is 'supported' in Ω. Then semiclassical techniques are used to estimate $MR_0(z, \theta)$ which involves the size of the Green function in the forbidden region. It is typical to obtain estimates like

$$h^2 \| J_i' p (z - H_1(\theta))^{-1} \| \to 0$$

as $h \to 0$. However, more refined estimates are possible and are required in the Stark ladder case. This is because the quantities of interest are exponentially small. A typical estimate on the Green function of $H_1(\theta)$ and H_2 localised to Ω and for z near e_0 takes the form:

$$\|\chi_\Omega R_i(z)\chi_\Omega\| \leqslant 1 + c\, e^{-2d_\Omega}\|R_i(z)\| \qquad i = 1, 2$$

where χ_Ω is a smooth characteristic function for Ω ($\chi_\Omega = 1$ on Ω and vanishes in a small neighbourhood outside Ω) and d_Ω is the distance in the Agmon metric (6) from Ω to $\mathbb{R}\backslash G(e_0)$, the complement of the forbidden region. These estimates together with the geometric resolvent equation are used to prove theorem 1.

With regard to theorem 2, we note that it is a more refined estimate than what appears in theorem 1. The estimate in theorem 1 involves the distance in the Agmon metric across the last, exit barrier which is relatively insensitive to F. The result of theorem 2 is obtained by iterating the geometric resolvent equation across the non-resonant cells. The non-resonant character of these cells is used to obtain a basis of functions which are exponentially well localised to the resonant cells only.

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